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Singular Operators on Boson Fields as Forms on Spaces of Entire Functions on Hilbert Space

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Invariant scales of entire analytic functions on Hilbert space are introduced and applied. Singular operators represented by sesquilinear forms on spaces of regular vectors are given explicit integral representations via kernels that are entire functions on the direct sum of the Hilbert space with its dual. The Weyl (or, exponentiated boson field) operators act smoothly and irreducibly on corresponding spaces of entire functions. Arbitrary symplectic operators on a single-particle Hilbert space are shown to be implementable on the corresponding boson field by appropriate generalized operators.

1. INTRODUCTION

A variety of singular operators arise in quantum field theory, of such disparate character that the establishment of an effective domain may become in large part the crux of the issue. The fine tuning required for the selection of function spaces involved in the treatment of nonlinear partial differential equation is familiar in the case of classical equations (i.e., those in which the values of the unknown function are substantially numerical). But the same is perhaps even more true in the case of quantized nonlinear partial differential equations, in which the unknown function is not only of a generalized character (e.g., distributional) but has values that are linear operators on an infinite-dimensional space.

For example, the treatment of quantized scalar nonlinear wave equations in two space–time dimensions involves a Hamiltonian H that is given formally as the sum $H_0 + V$ of "free" and "interaction" Hamiltonians H_0 and V. There is no manifest common domain, such as an appropriate Sobolev space will often provide for classical equations. The analytically controlled treatment of this problem has involved the introduction of a scale of

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L_p-spaces that is neither temporally invariant nor, as yet, effective in the treatment of higher-dimensional problems in quantum field theory.

This paper initiates a study of invariant scales that have the potential to be more broadly applicable than the cited L_p-scale. More specifically we treat here scales of spaces of entire functions on Hilbert space, in connection with the complex wave representation for boson fields (Segal [5]). This provides a natural and powerful format in which to treat the singular operators that arise in quantum field theory, and indeed it has been applied by one of us (Zhou [9]) to the development of a temporally invariant L_p-scale of entire functions on Hilbert space, which remarkably enjoys the contractability properties of the earlier real L_p-scale, although the two are fundamentally distinct (except when $p=2$). Here we focus on the scale of spaces S_t consisting of the domains of the unbounded operators e^{tH}, where H is the given basic Hamiltonian, which is effectively coherent with the complex analytic format.

Generalized operators, as represented by continuous sesquilinear forms on corresponding domains of entire functions, are shown to have quite convenient unique and appropriately domained kernel representations by entire functions. The Weyl (or exponentiated "field") operators $W(z)$ are shown to act smoothly and irreducibly on scales of entire functions. This raises the question of the corresponding cohomology of the action of the infinite-dimensional Heisenberg group that is defined by the Weyl relations, providing a promising basis for the rigorous correlation of formal local products of quantized fields with corresponding cocycles, which it is planned to investigate elsewhere.

Here we exemplify the "practical" use of the present scales by treating the important question of the implementability of given symplectic transformations on a single-particle Hilbert space, on the corresponding quantized boson field. Such a transformation S is known to be implementable on the field Hilbert space $\mathbf{K}$, in the sense that $TW(z)T^{-1}=W(Sz)$ for some unitary operator T on $\mathbf{K}$, and all z in the single-particle space $\mathbf{H}$, if and only if the commutator $[i, S]$ is Hilbert–Schmidt on the real Hilbert space $\mathbf{H}^{\#}$ underlying $\mathbf{H}$ (Segal [4], Shale [6]). In practice the Hilbert–Schmidt constraint is rarely satisfied, but formal versions of T abound, notwithstanding the rigorous impossibility of their existence as unitary operators on $\mathbf{K}$. We show here that this practice can be justified by the "projective" implementability, which is roughly implementability modulo multiplicative constants, of arbitrary symplectics. In precise terms, there is a unique generalized operator T such that $TW(z)=W(Sz)T$ and which satisfies the non-triviality and normalization condition, $\langle Tv, v\rangle = 1$, where v is the vacuum vector. A by-product is an explicit form for the (global) cocycle associated with the projective ("harmonic") representation of the unitarily implementable subgroup.

2. Technical Preliminaries

The following notation will be used, unless otherwise indicated. **H** will denote a given complex Hilbert space. $(\mathbf{K}, W, \Gamma, v)$ will denote the free boson field over **H**, in the sense of Segal [5]. Thus, **K** is a complex Hilbert space; W is a map from **H** to unitary operators on **K** satisfying the Weyl relations; Γ is a unitary representation on **K** of the full unitary group on **H**, which intertwines appropriately with W; and v is a unit vector in **K** that is invariant under all $\Gamma(U)$ and is cyclic for the $\{W(z) : z \in \mathbf{H}\}$. For any self-adjoint operator A in **H**, the self-adjoint generator of the one-parameter unitary group $\Gamma(e^{itA})$ will be denoted as $\partial\Gamma(A)$.

If R is a given operator in a Banach space **B**, the intersection of the domains $D(R^n)$, $n = 1, 2, ...,$ will be denoted as $D_\infty(R)$. This set will be denoted as $\mathbf{D}_\infty(R)$ when topologized sequentially, with the convergence of a sequence u_n to a vector u defined to mean that $R^m u_n \to R^m u$ in **B** for all $m = 1, 2, ...$ The set of all analytic (resp. entire) vectors (cf., e.g., Goodman [2]) for R in **B** will be denoted as $A(R)$ (resp. $E(R)$). When this set is topologized in the natural way so that convergence of a sequence u_n to u means that $e^{tR} u_n \to e^{tR} u$ for all sufficiently small t (resp. for all t), in **B**, this set will be denoted as $\mathbf{A}(R)$ (resp. $\mathbf{E}(R)$).

If **M** is a given real Hilbert space, the centered Gaussian measure of covariance operator C will be denoted as ν_C. When **M** is infinite-dimensional, ν_C is not in general countably additive but assigns a unique *integral* or *expectation* $E(p)$ to polynomials p over **H**, where such a polynomial is defined as a function of the form $p(x) = F(x_1, ..., x_n)$, where F is a polynomial on $\mathbb{R}^n$ and $x_j = \langle x, e_j \rangle$ for some finite set of orthonormal vectors $e_1, ..., e_n$ in $D(C^{1/2})$. We also write $E(p) = \int_{\mathbf{M}} p(x)\, d\nu_C(x)$ but note once more that the measure here need not be countably additive, so that the integral is not necessarily of the usual Lebesgue type. Thus if $C = I$,

$$E(p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} F(x_1, ..., x_n) \exp[-\tfrac{1}{2}\langle x, x \rangle]\, dx_1 \cdots dx_n$$

$$= \int_{\mathbf{M}} p(x)\, d\nu_I(x).$$

The set of all polynomials on **M** will be denoted as **P**. The completion of **P** in the L_p-norm, $\|f\|_p = E(|f|^p)^{1/p}$, where E and $\int_{\mathbf{M}}$ are extended in the obvious way to general Borel functions of $x_1, ..., x_n$, will be denoted as $L_p(\mathbf{M}, \nu)$, for $p \in [1, \infty)$.

As a real Hilbert space whose inner product is $\mathrm{Re}(\langle \cdot, \cdot \rangle)$, the complex Hilbert space **H** will be denoted as $\mathbf{H}^\#$. In $\mathbf{H}^\#$ we use, unless otherwise indicated, the covariance operator $C = \frac{1}{2}I$. Thus for any $p \in \mathbf{P}(\mathbf{H}^\#)$, and any orthonormal set $e_1, ..., e_n$ in $\mathbf{H}$ such that $p(z) = F(x_1, ..., x_n, y_1, ..., y_n)$, where $\langle z, e_j \rangle = x_j + iy_j$ and F is a polynomial on $\mathbb{R}^{2n}$,

$$E(p) = \pi^{-n} \int_{\mathbb{R}^{2n}} F(z) \exp[-\|z\|^2] \, dx_1 \cdots dx_n \, dy_1 \cdots dy_n.$$

The polynomials on $\mathbf{H}^{\#}$ that are holomorphic as functions on $\mathbf{H}$ form a subset that will be denoted as $\mathbf{P}^+(\mathbf{H})$; those that are antiholomorphic will be denoted as $\mathbf{P}^-(\mathbf{H})$. The closure in $L_p(\mathbf{H}^{\#})$ (where the measure is understood to be $\nu_{I/2}$ unless otherwise indicated) of $\mathbf{P}^{\pm}(\mathbf{H})$ will be denoted as $\mathbf{H}^{\pm} L_p(\mathbf{H})$. E can be extended by continuity to all of these spaces.

We define an entire function on a complex topological vector space $\mathbf{V}$ as one whose restriction to every finite-dimensional subspace is an entire analytic function in the usual sense, and we denote the set of all such functions on $\mathbf{V}$ as $\mathbf{H}^+(\mathbf{V})$. An antientire function on $\mathbf{V}$ is defined as one whose complex conjugate is entire, and the set of all such on $\mathbf{V}$ will be denoted as $\mathbf{H}^-(\mathbf{V})$. We recall the following aspects of the representation of the space $\mathbf{H}^{\pm} L_2(\mathbf{H})$ as subspaces of $\mathbf{H}^{\pm}(\mathbf{H})$.

(1) If $f \in \mathbf{H}^{\pm}(\mathbf{H})$, then $f \in \mathbf{H}^{\pm} L_2(\mathbf{H})$ if and only if the supremum over all finite-dimensional subspaces $\mathbf{M}$ of $\|f \mid \mathbf{M}\|_2$ is finite, and $\|f\|_2$ is then equal to this supremum.

(2) Let $e_1, e_2, ...$ be an orthonormal basis in $\mathbf{H}$. Then $F \in \mathbf{H}^+ L_2(\mathbf{H})$ if and only if there exist complex numbers a_n, where n represents a multi-index $(n_1, n_2, ...)$ with n_k a non-negative integer for all k such that $n_k = 0$ for sufficiently large k (which hereafter will be called simply a multi-index), such that $F(z) = \sum_n a_n z^n$, where

$$z^n = z_1^{n_1} z_2^{n_2} \cdots, \qquad z_j = \langle z, e_j \rangle;$$

and $\|F\|_2^2 = \sum_n n! \, |a_n|^2 < \infty$, where $n! = n_1! \, n_2! \cdots$.

(3) The free boson field over $\mathbf{H}$ is unitarily equivalent to the following:

(i) $\mathbf{K} = \mathbf{H}^- L_2(\mathbf{H})$.

(ii) For arbitrary $z \in \mathbf{H}$, $W(z)$ acts as follows on $f \in \mathbf{K}$: $f \to f_z$, where (with $\sigma = 1/\sqrt{2}$ here and henceforth)

$$f_z(u) = f(u - \sigma z) \exp(\sigma \langle z, u \rangle - \langle z, z \rangle / 4).$$

(iii) For an arbitrary unitary operator T on $\mathbf{H}$, $\Gamma(T)$ is the operator $f(u) \to f(T^{-1} u)$, $f \in \mathbf{K}$.

(iv) v is the function identically 1 on $\mathbf{H}$.

(4) If $F \in \mathbf{H}^- L_2(\mathbf{H})$, then $F(z) = \langle F, e_z \rangle$, where $e_z(u) = e^{\langle z, u \rangle}$ and $|F(z)| \leqslant e^{|z|^2/2} \|F\|_2$.

Our general usage regarding duality in vector spaces is as follows. The dual of the topological vector space **V** is denoted as **V***, and the antidual (or space of antilinear functions) as ***V**. In the case of a complex Hilbert space **H**, we have occasion to use the canonical correspondences between **H** on the one hand and **H*** and ***H** on the other, in which to the vector x in **H** corresponds the linear functional λ_x, where $\lambda_x(y) = \langle y, x \rangle$, and the antilinear functional ${}_x\lambda(y) = \langle x, y \rangle$. If F is an entire function on **H**, the function F^* on **H***, $F^*(\lambda_x) = F(x)$, $x \in \mathbf{H}$, is antientire, and vice versa; F^* is called the transfer of F from **H** to **H***.

Now let **V** be a topological vector space and J a continuous linear isomorphism of **V** onto a dense subspace of the Hilbert space **H**. It will avoid circumlocution to identify **V** with a subspace of **H** having an intrinsic stronger topology, leading in particular to the chain $\mathbf{V} \subset \mathbf{H} \cong {}^*\mathbf{H} \subset {}^*\mathbf{V}$, providing a canonical linear embedding of **V** into ***V**. If T is a linear operator in **H** that is moreover defined and continuous on **V** into itself, there is a corresponding linear operator ${}^*T$ in ***V** defined by the equation $({}^*Tf)(x) = f(T^*x)$, $x \in \mathbf{V}$, $f \in {}^*\mathbf{V}$, where T^* denotes the usual Hilbert space adjoint of T, provided T^* is defined on and leaves **V** invariant. In particular, if H is a given self-adjoint operator in **H**, the operators e^{sH} act continuously on $\mathbf{E}(H)$, for arbitrary $s \in \mathbb{C}$, and so may be extended by duality to operate also on ${}^*\mathbf{E}(H)$. We topologize ${}^*\mathbf{E}(H)$ by defining the convergence of a sequence u_n in ${}^*\mathbf{E}(H)$ to mean the convergence in **H** of the sequences $e^{-tH}u_n$, for all sufficiently large t. For the antidual ${}^*\mathbf{A}(H)$ to the space of analytic vectors for H, convergence is defined as the convergence of the sequence $e^{-tH}u_n$ in **H**, for some $t > 0$.

An equivalent, more concrete (but less manifestly invariant) formalism derives from spectral theory, according to which **H** may be regarded as $L_2(M)$ for some measure space M, in such a way that the given self-adjoint operator H is represented by the operation of multiplication by the real measurable function h, which we assume for simplicity to be positive. $\mathbf{E}(H)$ then consists of all functions f in $L_2(M)$ such that $e^{th}f$ is in $L_2(M)$ for all real t; ${}^*\mathbf{E}(H)$ consists of all measurable functions f such that $e^{-th}f \in L_2(M)$ for all sufficiently large t. Duality is given by the inner product $\langle f, g \rangle = \int_M f\bar{g}$, where the integrand will be integrable provided $f \in \mathbf{E}(H)$ and $g \in {}^*\mathbf{E}(H)$. In $\mathbf{E}(H)$, convergence of the sequence u_n to u means that $\|e^{th}(u_n - u)\| \to 0$ as $n \to \infty$ for all $t > 0$, while in ${}^*\mathbf{E}(H)$, this means that $\|e^{-th}(u_n - u)\| \to 0$ as $n \to \infty$ for all sufficiently large t. Similarly in the case of the space of analytic vectors, which is represented by the space of all measurable functions f on M such that $e^{th}f \in L_2(M)$ for sufficiently small $t > 0$, and its antidual, represented by those such that $e^{-th}f \in L_2(M)$ for some $t > 0$, and analogous convergence criteria and duality.

3. Kernels of Generalized Operators on Boson Fields

Throughout this section, B will be a given self-adjoint operator in the Hilbert space $\mathbf{H}$ assumed for simplicity and applicability to be strictly positive, meaning positive and bounded away from 0. H will denote the operator $\partial\Gamma(B)$ in $\mathbf{K}$, where $(\mathbf{K}, W, \Gamma, v)$ is the free boson field over $\mathbf{H}$. Generalized operators will be treated in the format of continuous sesquilinear forms on subspaces of regular vectors in $\mathbf{K}$, notably the subspaces $\mathbf{E}(H)$ and $\mathbf{A}(H)$.

In this context, we define an *entire kernel* as a function $K(z, z')$ defined on $\mathbf{E}(B) \times \mathbf{E}(B)$ that is antientire as a function of z, and entire as a function of z', and has the property that for all sufficiently large $t > 0$, $K(e^{-tB}z, e^{-tB}z')$ is in $L_2(\mathbf{H} \oplus \mathbf{H})$. An *analytic kernel* is defined similarly except that $K(e^{-tB}z, e^{-tB}z')$ is required to be in $L_2(\mathbf{H} \oplus \mathbf{H})$ for all $t > 0$. In applications to the quantization of wave equations on compact manifolds, the relevant operator B (so-called single-particle Hamiltonian) has the property that e^{-tB} is of trace class for all $t > 0$, and this property will play a key role in this section.

A *standard kernel* on a complex vector space $\mathbf{V}$ will be defined as a function $K(z, z')$ on $\mathbf{V} \times \mathbf{V}$ that is in $\mathbf{H}^-(\mathbf{V})$ for fixed z', and in $\mathbf{H}^+(\mathbf{V})$ for fixed z; or equivalently, with the natural transfers indicated earlier, in $\mathbf{H}^-(\mathbf{V} \oplus \mathbf{V}^*)$ or in $\mathbf{H}^+(\mathbf{V}^* \oplus \mathbf{V})$, if $\mathbf{V}$ is a Hilbert space.

Theorem 1. *Let B be a given positive self-adjoint operator in the Hilbert space $\mathbf{H}$, and let $H = \partial\Gamma(B)$ be the corresponding operator on the free boson field over $\mathbf{H}$. Let ϕ be a given continuous sesquilinear form on $\mathbf{E}(H)$. Suppose that e^{-tB} is of trace class for all $t > 0$.*

Then there exists a unique entire kernel $K(z, z')$ on $\mathbf{E}(B)$ such that

$$\phi(f, g) = \int_{\mathbf{H}} \int_{\mathbf{H}} K(z, z') f(z') \, d\nu(z') \, \bar{g}(z) \, d\nu(z) \tag{1}$$

or the iterated integral in the opposite order. Specifically,

$$K(z, z') = \phi(e_z, e_{z'})$$

and the inequality

$$|K(z, z')| \leqslant C \exp[\|e^{sB}z\|^2 + \|e^{sB}z'\|^2] \tag{2}$$

is satisfied for some $s > 0$.

Conversely, if $K(z, z')$ is a standard kernel on $\mathbf{E}(B)$ that satisfies inequality (2), *then there exists a continuous sesquilinear form ϕ on $\mathbf{E}(H)$ such that Eq.* (1) *holds.*

LEMMA 1.1. *Let $\mathbf{L}_n$ be an increasing sequence of finite-dimensional subspaces of the Hilbert space* $\mathbf{H}$, *whose union* $\mathbf{L}$ *is dense in* $\mathbf{H}$. *Let F be a given complex-valued function on* $\mathbf{L}$ *with the properties*: (i) $F|\mathbf{L}_n$ *is entire for all n*; (ii) $\int_{\mathbf{L}_n} |F(z)|^2\, dv(z)$ *is bounded as $n \to \infty$. Then F has a unique extension to an entire function on all of* $\mathbf{H}$, *which is in* $\mathbf{H}^+ L_2(\mathbf{H})$.

Proof. If m is a multi-index and $z \in \mathbf{H}$, we define z^m, relative to a given orthonormal basis $e_1, e_2, \ldots$ for $\mathbf{H}$, as $z_1^{m_1} z_2^{m_2} \cdots$, for $z = \sum z_j e_j$, $z \in \mathbf{H}$. Let $F_n = F|\mathbf{L}_n$; then for $z \in \mathbf{L}_n$, $F_n(z) = \sum_m a_m z^m$, the summation being over m-indices such that $m_k = 0$ for $k > n$; the basis $e_1, e_2, \ldots$ for $\mathbf{H}$ is chosen so that $e_1, e_2, \ldots, e_n$ is a basis for $\mathbf{L}_n$. Moreover, $\sum_m m!\, |a_m|^2 = \int_{\mathbf{L}_n} |F_n(z)|^2\, dv(z)$, where the suummation is over those multi-indices for which $m_k = 0$ when $k > n$. If $n' > n$, the coefficients $a'_{m'}$ for $F_{n'}$ involve multi-indices m' such that $m'_k = 0$ for $k > n'$. But since $a_m = \langle F_n, z^m \rangle / m!$ for all m, the a'_m for a multi-index m with $m_k = 0$ for $k > n$ coincides with the a_m. It follows that $\sum_m m!\, |a_m|^2$, where the sum is over all multi-indices m, without restriction, is convergent. It follows in turn that the series $\sum_m a_m z^m$ is convergent for all $z \in \mathbf{H}$, and this series defines the extension claimed. Any extension will have the same coefficients a_m relative to the basis z^m, so the extension is unique. ∎

LEMMA 1.2. *Let B be a positive self-adjoint operator in* $\mathbf{H}$ *such that e^{-tB} is a trace class operator for some real t. Let $H = \partial\Gamma(B)$ be the corresponding operator on* $\mathbf{K}$. *Then e^{-tH} is a trace class operator, and*

$$\operatorname{tr}(e^{-tH}) = \prod_j (1 - e^{-\lambda_j t})^{-1},$$

where λ_j are eigenvalues of B.

Proof. Since the boson field over the direct sum of Hilbert spaces is the tensor product of the boson fields over the constituents, $\mathbf{K}$ may be represented as the tensor product of the boson fields over the one-dimensional subspaces $\mathbf{H}_j$ of $\mathbf{H}$ spanned by the e_j. In $\mathbf{H}_j$, B acts as λ_j, and H has the eigenvalues $n\lambda_j$, where $n = 0, 1, 2, \ldots$. Thus

$$\operatorname{tr}(e^{-tH}) = \prod_j \sum_{n=0}^{\infty} e^{-tn\lambda_j} = \prod_j (1 - e^{-\lambda_j t})^{-1}.$$

The infinite product indicated is convergent provided $\sum_j e^{-\lambda_j t}$ is convergent, as assumed, and the lemma follows. ∎

LEMMA 1.3. *Suppose that e^{-tB} is a trace class operator for all $t > 0$, B being a positive self-adjoint operator in* $\mathbf{H}$, *and let $H = \partial\Gamma(B)$. Then the antidual of the space* $\mathbf{E}(H)$ *of entire vectors for H is representable as follows.*

Let $\mathbf{F}$ *denote the space of all antientire functions on the space* $\mathbf{E}(B)$ *that are of the form*

$$f(z)=\sum_n b_n \bar{z}^n; \qquad \sum_n n!\,|b_n|^2\,e^{-tn\lambda}<\infty$$

for sufficiently large t, *where* $\bar{z}^n=\bar{z}_1^{n_1}\bar{z}_2^{n_2}\cdots$, $z_j=\langle z, e_j\rangle$, *the* e_j *forming an orthonormal basis for* $\mathbf{H}$ *such that* $Be_j=\lambda_j e_j$, *and* $e^{-tn\lambda}$ *denotes* $e^{-t(n_1\lambda_1+n_2\lambda_2+\cdots)}$. *Then for arbitrary* $f\in\mathbf{F}$, *the functional* ψ *on* $\mathbf{E}(H)$ *defined by the equation*

$$\psi(g)=\sum_n n!\,\overline{a_n}b_n; \qquad g(z)=\sum_n a_n\bar{z}^n,$$

exists and is in $^*\mathbf{E}(H)$, *and every vector in* $^*\mathbf{E}(H)$ *is of this form for a unique vector* $f\in\mathbf{F}$.

Proof. Let the canonical pairing between $\mathbf{E}(H)$ and $^*\mathbf{E}(H)$ be denoted as $\langle\psi, g\rangle$ $(g\in\mathbf{E}(H),\ \psi\in{}^*\mathbf{E}(H))$. Then by the definition of the action of e^{-tH} in $^*\mathbf{E}(H)$, $\langle\psi, g\rangle=\langle e^{-tH}\psi, e^{tH}g\rangle$. If t is sufficiently large, $e^{-tH}\psi$ will be in $\mathbf{K}$, whence $e^{-tH}\psi$ is representable by a vector $h\in\mathbf{H}^-L_2(\mathbf{H})$ of the form $h(z)=\sum_n c_n\bar{z}^n$ with $\sum_n n!\,|c_n|^2<\infty$. Defining $f(z)=h(e^{tB}z)$ for $z\in\mathbf{E}(B)$, the lemma follows. ∎

LEMMA 1.4. *With the identification of* $^*\mathbf{E}(H)$ *with the space of antientire functions given by Lemma* 1.3, *the action of* e^{tH} *on* $^*\mathbf{E}\ (H)$ *may be extended as follows: if* $f\in\mathbf{F}$ *corresponds to the vector* $w\in{}^*\mathbf{E}(H)$, *then to the vector* $e^{tH}w$ *in* $^*\mathbf{E}(H)$ *corresponds the function* $f_t\in\mathbf{H}^-(\mathbf{E}(B))$ *given by the equation*

$$f_t(z)=f(e^{tB}z).$$

Proof. If f is a monomial in the z_j, the conclusion of the lemma is immediate. For general f, the conclusion follows from this by linearity and continuity. ∎

LEMMA 1.5. *Let* $K(z,z')$ *be a standard kernel that is in* $L_2(\mathbf{H}\oplus\mathbf{H})$. *Then the operator* K *on* $\mathbf{H}^-L_2(\mathbf{H})$ *given by the equation*

$$(Kf)(z)=\int_{\mathbf{H}} K(z,z')\,f(z')\,dv(z')$$

is Hilbert–Schmidt. Moreover, its Hilbert–Schmidt norm $\|K\|_2$ *is given by the equation* $\|K\|_2=\|K(\cdot,\cdot)\|_{L_2(\mathbf{H}\oplus\mathbf{H})}$.

Conversely, given a Hilbert–Schmidt operator K *on* $\mathbf{H}^-L_2(\mathbf{H})$, *there exists a unique standard kernel* $K(z,z')$ *such that the foregoing holds. This is given by the equation* $K(z,z')=\langle Ke_z, e_{z'}\rangle$.

Proof. Consider, to begin with, the case in which $K(z, z')$ is a given polynomial, antiholomorphic in z and holomorphic in z'. Observe that the inner product in $L_2(\mathbf{H})$ of two monomials of which one is holomorphic and the other antiholomorphic, and which are not both of degree 0, vanishes. It follows that if K is the operator whose kernel is $K(z, z')$, then

$$\|K\|_2^2 = \sum_{m,n} |\langle Ke^m, e^n\rangle|^2,$$

where m and n are multi-indices, and $e^m = \bar{z}^m/\sqrt{m!}$, since all cross terms $\langle Kf, g\rangle$, where either f or g is not antiholomorphic, will vanish, by virtue of the fact that the range of K consists only of antiholomorphic functions. But the right side of the preceding equality is just the squared L_2-norm, $\|K(\cdot, \cdot)\|^2_{L_2(\mathbf{H}\oplus\mathbf{H})}$. Since the indicated polynomials are dense in the space of all square-integrable standard kernels (which as noted earlier can be identified with $\mathbf{H}^+L_2(\mathbf{H}^*\oplus\mathbf{H})$), the first part of Lemma 1.5 now follows by continuity.

If conversely K is a given Hilbert–Schmidt operator on $\mathbf{H}^-L_2(\mathbf{H})$, it has a kernel in $L_2(\mathbf{H}^*\oplus\mathbf{H})$ whose projection onto $\mathbf{H}^+L_2(\mathbf{H}^*\oplus\mathbf{H})$ is the kernel having the properties claimed in the lemma (using property (4) in Section 2). ∎

LEMMA 1.6. *If F is entire on* $\mathbf{H}$ *and if* $|F(z)| \leqslant C\exp(\frac{1}{2}\|Sz\|^2)$ *for some Hilbert–Schmidt operator S on* $\mathbf{H}$ *with* $\|S\| < 1$, *then* $F \in \mathbf{H}^+L_2(\mathbf{H})$.

Proof. It is no essential loss of generality to assume that S is positive and self-adjoint. Let $\{e_j\}$ be an orthonormal basis in $\mathbf{H}$ such that $Se_j = s_j e_j$, where $s_j > 0$. Let $\mathbf{H}_n$ be the linear manifold spanned by $e_1, ..., e_n$. It suffices to show that $\int_{\mathbf{H}_n} |F(z)|^2\, dv(z)$ is bounded as $n \to \infty$. The integral over $\mathbf{H}_n$ is in fact bounded by C^2 times the product ($j = 1, ..., n$) of the one-dimensional integrals $\int_{\mathbf{C}} \exp(s_j^2\, |z|^2)\ dv(z) = (1 - s_j^2)^{-1}$. The infinite product of the latter is convergent provided the series $\sum_j s_j^2$ is convergent, as is implied by the Hilbert–Schmidt character of S. ∎

DEFINITION. An entire function F on a dense subspace $\mathbf{D}$ of the Hilbert space $\mathbf{H}$ is said to be *effectively in* $\mathbf{H}^+L_2(\mathbf{H})$ in case $\int_{\mathbf{L}} |F(z)|^2\, dv(z)$ is bounded as $\mathbf{L}$ ranges over the finite-dimensional subspaces of $\mathbf{D}$. The argument of Lemma 1.1 then shows that F admits a unique extension to an entire function on all of $\mathbf{H}$ that is in $\mathbf{H}^+L_2(\mathbf{H})$.

In the context of a given positive self-adjoint operator B in a Hilbert space $\mathbf{H}$, inner products of vectors in the antidual $^*\mathbf{E}(B)$ of the space of entire vectors $\mathbf{E}(B)$ will be partially defined by continuity. Thus, if x and y are in $^*\mathbf{E}(B)$, the inner product $\langle x, y\rangle$ is defined in case $x \in \mathbf{D}(e^{sB})$ and $y \in \mathbf{D}(e^{tB})$ with $s + t \geqslant 0$ as the inner product in $\mathbf{H}$, $\langle e^{-(s+t)B}x', y'\rangle$, where

$x' = e^{sB}x$ and $y = e^{tB}y$. As thus defined, $\langle x, y\rangle$ is independent of the particular values of s and t, as long as the indicated conditions are satisfied, as is clear from the formulation of the inner product in terms of the spectral representation of B as a multiplication by a measurable function h on an L_2-space.

Since functional integration over an infinite-dimensional space in the present sense is not a form of Lebesgue integration, the Fubini theorem does not apply. Results similar to some of its consequences are however obtainable via the underlying holomorphy of the functions involved.

LEMMA 1.7. *Let $K(z, z')$ be a standard kernel that is square integrable, regarded as an entire function on $\mathbf{H}^* \oplus \mathbf{H}$. Let f and g be arbitrary in $\mathbf{H}^- L_2(\mathbf{H})$. Then*

(1) *For any fixed z in $\mathbf{H}$, $K(z, \cdot)$ is in $\mathbf{H}^+ L_2(\mathbf{H})$ and $K(\cdot, z')$ is in $\mathbf{H}^- L_2(\mathbf{H})$.*

(2) *Let $h(z)$ denote the inner product $\langle K(z, \cdot), \bar{f}\rangle_{\mathbf{H}^+ L_2(\mathbf{H})}$. Then $h(\cdot)$ is in $\mathbf{H}^- L_2(\mathbf{H})$.*

(3) $\langle h, g\rangle_{\mathbf{H}^- L_2(\mathbf{H})} = \langle K, g \otimes \bar{f}\rangle_{\mathbf{H}^+ L_2(\mathbf{H}^* \oplus \mathbf{H})} = \langle k, \bar{f}\rangle$, *where* $k(z') = \langle K(\cdot, z'), g\rangle_{\mathbf{H}^- L_2(\mathbf{H})}$.

Proof. Parts (1)–(3) are immediate when $K(z, z')$ is a polynomial that is antiholomorphic in z and holomorphic in z', and when f and g are antiholomorphic polynomials. Passage to simultaneous L_2 limits is straightforward with the use of the properties given in Section 2. ∎

Completion of Proof of Theorem. Let ϕ_t denote the form given by the equation $\phi_t(f, g) = \phi(e^{-tH}f, e^{-tH}g)$. By the continuity of ϕ, there exists $t_0 > 0$ such that ϕ_{t_0} is bounded. Accordingly, if $t > t_0$, the form ϕ_t is Hilbert–Schmidt. By Lemma 1.5, there exists a square-integrable standard kernel $K_t(z, z')$ given by the equation (where we write $e_u = e(u)$)

$$K_t(z, z') = \phi_t(e_z, e_{z'}) = \phi(e(e^{-tB}z), e(e^{-tB}z')),$$

such that

$$\phi_t(f, g) = \iint K_t(z, z')\, f(z')\, \bar{g}(z)\, d\nu(z)\, d\nu(z').$$

Replacing f and g by $e^{tH}f$ and $e^{tH}g$, it follows that

$$\phi(f, g) = \iint K_t(z, z')(e^{tH}f)(z')\overline{(e^{tH}g)}(z)\, d\nu(z)\, d\nu(z').$$

Now setting $K(z, z') = \phi(e_z, e_{z'})$ for $z, z' \in \mathbf{E}(B)$, one may express the last integral as $\iint K(e^{-tB}z, e^{-tB}z')(e^{tH}f)(z')\overline{(e^{tH}g)}(z)\, dv(z)\, dv(z')$. On the other hand, for arbitrary entire functions F and G on $\mathbf{E}(B)$ with G effectively in $\mathbf{H}^+L_2(\mathbf{H})$ and indeed in $\mathbf{E}(H)$, and F effectively in $^*\mathbf{E}(H)$, the integral $\int F(e^{-tB}z)\overline{(e^{tH}G)}(z)\, dv(z)$ is independent of t and equal to $\int F(z)\, \bar{G}(z)\, dv(z)$, as an inner product of generalized vectors in $^*\mathbf{E}(H)$. When this is applied with B replaced by $^*B \oplus B$ in $^*\mathbf{H} \oplus \mathbf{H}$, it follows that the last integral equals $\iint K(z, z')\, f(z')\, \bar{g}(z)\, dv(z)\, dv(z')$. The inequality (2) then follows from (4) of Section 2 and the given expression for $K(z, z')$.

Now suppose that $K(z, z')$ is given having the indicated properties. Since e^{-tB} has absolutely convergent trace for all $t > 0$, the kernel $K_t(z, z') = K(e^{-tB}z, e^{-tB}z')$ is a Hilbert–Schmidt kernel for sufficiently large $t > 0$, where Lemma 1.6 is used. Let ϕ_t denote the sesquilinear form on $\mathbf{H}^-L_2(\mathbf{H})$ having kernel K_t, and set $\phi(f, g) = \phi_t(e^{tH}f, e^{tH}g)$. Then ϕ is the required form corresponding to $K(z, z')$. ∎

COROLLARY 1.1. *With the assumptions of Theorem* 1,

$$\iint K(e^{-sB}z, e^{-tB}z')(e^{sH}f)(z')\overline{(e^{tH}g)}(z)\, dv(z)\, dv(z')$$

is independent of s and t for non-negative s and t.

Proof. The same argument as that used in the case $s = t$ treated above is applicable. ∎

COROLLARY 1.2. *With the notation of Theorem* 1, $K(f, g)$ *is expressible as a limit of conventional Lebesgue integrals in the form*

$$K(f, g) = \lim_{n \to \infty} \iint_{\mathbf{H}_n} K(P_n z, P_n z')\, f(P_n z')\, \bar{g}(P_n z)\, dv(z)\, dv(z'),$$

where P_n *is the projection with range* $\mathbf{H}_n$, *and* $\mathbf{H}_n$ *is the subspace spanned by the first n proper vectors of B.*

Proof. The integral is, e.g., a limit of the integrals of the conditional expectations of the integrand with respect to an ascending sequence of sigma-rings whose union generates the full sigma-ring of the underlying probability space, modulo null sets. ∎

The argument of the proof of Theorem 1 may be adapted to the treatment of corresponding questions for the space of analytic, rather than entire, vectors for H. More specifically, we may state

COROLLARY 1.3. *Let B, H, and $\mathbf{H}$ be as in Theorem* 1, *and let ϕ be a given continuous sesquilinear form on $\mathbf{A}(H)$. Suppose that e^{-tB} is trace class for some $t > 0$.*

Then there exists a unique standard kernel $K(z, z')$ on $\mathbf{A}(B)$ such that

$$\phi(f, g) = \int_{\mathbf{H}} \left[\int_{\mathbf{H}} K(z, z') f(z') \, dv(z') \right] \bar{g}(z) \, dv(z) \tag{3}$$

(or the iterated integral in the opposite order). Specifically,

$$K(z, z') = \phi(e_z, e_{z'}),$$

and the inequality

$$|K(z, z')| \leqslant C \exp(\|e^{sB} z\|^2 + \|e^{sB} z'\|^2) \tag{4}$$

is satisfied for all $s > 0$.

Conversely, if $K(z, z')$ is a standard kernel on $\mathbf{A}(B)$ that satisfies the inequality (4), *then there exists a continuous sesquilinear form ϕ on $\mathbf{A}(H)$ such that Eq.* (3) *holds.*

Proof. The argument for Theorem 1 applies with the following modifications. In Lemma 1.3, the dual of $\mathbf{A}(H)$ is represented by the functions $f(z)$ of the given form except that $\sum_n |b_n|^2 \, n!^{-1} e^{-tn\lambda} < \infty$ for all $t > 0$, rather than only for sufficiently large t. In Lemma 1.4, the same expression for f_t is valid if f is required only to be in $^*\mathbf{A}(B)$.

In the final argument, ϕ_t must be bounded for all $t > 0$, and hence also Hilbert–Schmidt for all $t > 0$, by the Hilbert–Schmidt character of e^{-tH}. The inequality (4) for all $t > 0$ then follows in the same way as earlier. ∎

4. ACTION OF THE WEYL SYSTEM ON REGULAR VECTORS

The Weyl operators $W(z)$ form the basis for a unitary representation of the infinite-dimensional Heisenberg group on $\mathbf{K}$. As often happens in the theory of infinite-dimensional groups, issues arise requiring regularity greater than that provided by the continuity of the $W(z)$ in the pure Hilbert space sense. In particular questions arise in connection with the characterization of renormalized local products of quantum fields (e.g., Baez *et al.* [1]) that involve properties of the action on the spaces of regular vectors.

THEOREM 2. *With the same notation as that given earlier, the map $(z, f) \to W(z) f$ from $E(B) \times E(H)$ into $\mathbf{K}$ is into $E(H)$ and is continuous from $\mathbf{E}(B) \times \mathbf{E}(H)$ into $\mathbf{E}(H)$.*

The proof is by treatment of the one-dimensional, then the finite-dimensional, and finally the infinite-dimensional case.

LEMMA 2.1. *Let t be arbitrary in $(0, \infty)$ and z arbitrary in $\mathbb{C}$. Set $c = \exp(-t)$, let b be arbitrary in $(0, (c^{-1} - c)^{-1})$, and set $k = [1 - b(c^{-1} - c)]^{-1/2}$. Let* $\mathbf{H}$ *be $\mathbb{C}$ as a one-dimensional Hilbert space, and let $B = 1$. Then for $f \in E(H)$,*

$$\|\exp(tH)\, W(z) f\|_2 \leqslant \exp(\tfrac{1}{2}(c^{-1} - c)(c + b^{-1})\, |z|^2)\, k^2\, \|\exp((t + \kappa) H) f\|_2,$$

where $\kappa = \log k$.

Proof. With the definitions and expressions given above,

$$(\exp(tH)\, W(z) f)(u) = f(c^{-1}u - \sigma z) \exp(\sigma\langle z, c^{-1}u\rangle - \tfrac{1}{4}\langle z, z\rangle).$$

Thus

$$\|\exp(tH)\, W(z) f\|_2^2 = \int |f(c^{-1}u - \sigma z)|^2 \exp(2\sigma \operatorname{Re}\langle z, c^{-1}u\rangle - \tfrac{1}{2}\langle z, z\rangle)\, dv(u).$$

With the change of variables $u \to w + \sigma c z$, this expression becomes

$$\int |f(c^{-1}w)|^2 \exp(\tfrac{1}{2}(1 - c^2)\, |z|^2 - |w|^2 + (c^{-1} - c)\, \sigma^{-1} \operatorname{Re}\langle w, z\rangle)\, dw,$$

where $dw = dw_1\, dw_2$, $w = w_1 + iw_2$, the w_j being real.

Now for any $b \in (0, \infty)$, $\operatorname{Re}\langle v, z\rangle \leqslant \tfrac{1}{2}[b\sigma^{-1}\, |v|^2 + b^{-1}\sigma\, |z|^2]$. Accordingly, the foregoing integral is bounded by the product of the function of z, $\exp(\tfrac{1}{2}(1 - c^2 + b^{-1}(c^{-1} - c))\, |z|^2)$, which equals $\exp(\tfrac{1}{2}(c^{-1} - c)(c + b^{-1})\, |z|^2)$, and the integral

$$\int |f(c^{-1}w)|^2 \exp(-|w|^2/2p^2)\, dw,$$

where p is given by the equation $(2p^2)^{-1} = 1 - (c^{-1} - c)b$ and is positive provided $b \in (0, (c^{-1} - c)^{-1})$, as assumed in the hypothesis of the lemma. Noting that $p\sigma^{-1} = k$, it follows that the last integral may be expressed as

$$k^2 \int |f(c^{-1}kv)|^2 \exp(-|v|^2)\, dv = k^2\, \|\exp((t + \kappa) H) f\|_2^2. \quad \blacksquare$$

LEMMA 2.2. *Suppose* $\mathbf{H}$ *is finite-dimensional. Let t be arbitrary in $(0, \infty)$ and z be arbitrary in* $\mathbf{H}$. *Set $C = \exp(-tB)$, where B is as earlier, and let D*

be a positive self-adjoint operator in $\mathbf{H}$ *that commutes with* B *and is* $\leqslant (C^{-1}-C)^{-1}$. *Let* K *denote* $(I-D(C^{-1}-C))^{-1/2}$. *Then for* $f \in E(H)$,

$$\|\exp(tH)\, W(z) f\|_2 \leqslant \exp(\tfrac{1}{2}\langle (C^{-1}-C)(C+D^{-1})z, z\rangle)(\det K)^2\, \|\exp((t+\kappa)H) f\|_2,$$

where $\kappa = \|\log K\|$ (*maximal eigenvalue of* $\log K$).

Proof. Note first that Lemma 2.1 is unchanged if B is the constant matrix aI, and $c=\exp(-ta)$. Now if B is diagonalized in the general case and Lemma 2.1 is applied to each component, the present lemma follows. ∎

LEMMA 2.3. *With the same notation and assumptions as those in Lemma* 2.2 *except that* $\mathbf{H}$ *may be infinite-dimensional,*

$$\|\exp(tH)\, W(z) f\|_2 \leqslant \exp(\tfrac{1}{2}\langle C^{-3}z, z\rangle)(\det K)^2\, \|\exp((t+\kappa)H) f\|_2 \qquad (z \in E(B)).$$

Proof. With $D=C^2$, $(C^{-1}-C)(C+D^{-1}) = I - C^2 + C^{-3} - C^{-1} \leqslant C^{-3} + (I - C^{-1}) \leqslant C^{-3}$. Lemma 2.3 then follows from Lemma 2.2 when $\mathbf{H}$ is finite-dimensional. In the infinite-dimensional case, $K=(I-C+C^3)^{-1/2}$, which differs from I by a trace class operator, since $C-C^3$ is trace class. Since all eigenvalues of $C-C^3$ are less than 1, $\det K$ exists. Now let $\mathbf{H}_n$ be the subspace of $\mathbf{H}$ spanned by the first n eigenvectors of B, and suppose that z lies in some $\mathbf{H}_m$. The conclusion of the lemma then follows for vectors f that are supported by some $\mathbf{H}_m$, and hence by approximation for arbitrary $f \in \mathbf{E}(H)$. A further approximation argument eliminates the restriction that z lie in some $\mathbf{H}_m$ and permits it to be arbitrary in $\mathbf{E}(B)$. ∎

Proof of Theorem. It follows from Lemma 2.3 that if $z \in E(B)$ and $f \in E(H)$, then $W(z) f \in E(H)$. It remains only to show the continuity of the map $(z, f) \to W(z) f$, to which end we write

$$W(z) f - W(z') f' = (W(z) - W(z')) f + W(z')(f - f').$$

The second term on the right will be arbitrarily small if z' and f' are sufficiently close to z and f by Lemma 2.3. Consider therefore the first term on the right. By the Weyl relations and a simple estimate, consideration may be limited to the case $z'=0$. The term then takes the form, after application of $\exp(tH)$,

$$(\exp(tH)(W(z)-I) f)(u) = f(C^{-1}u - \sigma z)\exp(\sigma\langle C^{-1}z, u\rangle - \tfrac{1}{4}|z|^2) - f(C^{-1}u).$$

We need to show that this can be made arbitrarily small in $\mathbf{H}^{-}L_2(\mathbf{H})$ by restricting z to be sufficiently close to 0 in $\mathbf{E}(B)$ and f sufficiently close to f' in $\mathbf{E}(H)$. If the right side is represented as the sum of four terms in the usual way, this is implied by showing the same for each of the representative terms

$$f(C^{-1}u-\sigma z)-f(C^{-1}u), \qquad (\exp(\sigma\langle C^{-1}z, u\rangle - \tfrac{1}{4}|z|^2)-1)f(C^{-1}u).$$

For the first of these terms, this follows from the fact that if $h\in \mathbf{E}(H)$, then $h(u+z)\to h(u)$ in $\mathbf{H}^{-}L_2(\mathbf{H})$ as $z\to 0$ in $\mathbf{E}(B)$. That this is the case follows, e.g., by first considering the case of polynomials and then using approximation via the power series expansion of h. For the second term, this follows by an adaptation of the argument involving the Schwarz inequality in the proof of Lemma 2.1. ∎

COROLLARY 2.1. *The map $(z, f)\to W(z)f$ from $A(B)\times A(H)$ into* $\mathbf{K}$ *is in fact into $A(H)$ and is continuous from $\mathbf{A}(B)\times \mathbf{A}(H)$ into $\mathbf{A}(H)$.*

Proof. The proof is essentially the same except that in the final argument, $\exp(tH)(W(z)-I)f$ needs to be estimated only for sufficiently small $t>0$, on the basis of the related assumptions concerning z and f. These in turn are obvious variants of those given earlier. ∎

It will be important for the characterization of singular operators on boson fields via their transformation properties on conjugation by the $W(z)$ to establish the irreducibility of the action of $W(\cdot)$ on $E(H)$, in the generalized sense described in

THEOREM 3. *The only continuous sesquilinear forms K on $\mathbf{E}(H)$ such that*

$$K(W(z)u, W(z)u') = K(u, u')$$

for all $z\in \mathbf{E}(B)$ and $u, u'\in \mathbf{E}(H)$ are of the form $K(u, u') = c\langle u, u'\rangle$, where c is a constant.

Proof. This theorem differs from Theorem 3.2 of Segal [3] only in that the form is defined on $\mathbf{E}(H)$, rather than the space $\mathbf{D}_\infty(H)$ described. The proof is an adaptation of the proof of this earlier result, as follows. Let $\mathbf{Q}$ denote the space of all antiholomorphic polynomials $p(z)$ on $\mathbf{H}$, in the $\langle e_j, z\rangle$, where the e_j form an orthonormal basis of proper vectors for B. Then $\mathbf{Q}$ is dense in $\mathbf{K}$, is invariant under the $\exp(itH)$, $t\in\mathbb{R}$, and hence is dense in $\mathbf{E}(H)$ (by Lemma 3.5 of [3], or a direct argument). It suffices by the earlier argument to show that any continuous sesquilinear form on $\mathbf{Q}$, the continuity here being in sense of the relative topology induced from that in $\mathbf{E}(H)$, is a constant multiple of the inner product in $\mathbf{K}$, as restricted to $\mathbf{Q}$. If F is such a form, its restriction F_n to the subspace $\mathbf{Q}_n$ of $\mathbf{Q}$ consisting of polynomials in the $\langle e_j, z\rangle$ with $j \leqslant n$ is invariant under the $W(e_j)$ for $j \leqslant n$. The preceding theorem implies that $W(te_j)p$ is a differentiable function of t, with derivative $w(e_j)p$, which is in $\mathbf{Q}_n$ (where $w(z)$ denotes the self-adjoint generator of the one-parameter unitary group $W(tz)$, $t \in \mathbb{R}$). Invariance under the $W(e_j)$ of F_n then implies that $F_n(w(z)p, q) = F_n(p, w(z)q)$, for arbitrary $p, q \in \mathbf{Q}_n$ and $z = e_j$ for some $j \leqslant n$.

Now it is no essential loss of generality to suppose that $F(1, 1) = 0$, since otherwise $F(u, u')$ may be replaced by $F(u, u') - F(1, 1)\langle u, u'\rangle$, and with this assumption made, it follows from Lemma 3.6 of [3] that F_n vanishes identically. Hence F vanishes identically. ∎

COROLLARY 3.1. *If K is a continuous sesquilinear form on $\mathbf{A}(H)$ such that*

$$K(W(z)u, W(z)u') = K(u, u')$$

for all $z \in A(B)$ and $u, u \in A(H)$, then $K(u, u') = c\langle u, u'\rangle$ for some constant c.

Proof. The argument of the theorem extends with obvious modifications to the case of analytic rather than entire vectors. ∎

5. IMPLEMENTABILITY OF SYMPLECTIC TRANSFORMATIONS

In this section the foregoing treatment of singular operators on boson fields is applied to the important case of the induced action on a boson field of a given symplectic transformation on the underlying single-particle space $\mathbf{H}$. We recall that the symplectic group $Sp(\mathbf{H})$ consists of all real-linear bicontinuous transformations on $\mathbf{H}$ that preserve the form $\mathrm{Im}\langle z, z'\rangle$, z and z' being arbitrary in $\mathbf{H}$. If $S \in Sp(\mathbf{H})$ is such that $[S, i]$ is a Hilbert–Schmidt operator, then there exists a unitary transformation $\Gamma(S) = T$ on $\mathbf{K}$ such that $TW(z)T^{-1} = W(Sz)$ for all $z \in \mathbf{H}$ (Segal [4], Shale [6]). T is said to implement S. In physical practice the Hilbert–Schmidt condition is rarely satisfied in more than two space–time dimensions, and the Hilbert–Schmidt condition is necessary as well as sufficient for unitary implementability. Nevertheless, heuristic quantum field theory has long made use of symbolic transformations on boson fields that implement formally given symplectic transformations on an underlying unquantized (or single-particle) field space. Indeed, arbitrary symplectics are representable by automorphisms of a C^*-algebra generated by the Weyl operators $W(z)$, but the relation of the operators that occur in practice to this algebra is not always clear. In any event, detailed analysis is facilitated by a representation of the putative similarity transformation T implementing the automorphism, which is given here.

The transformation T is necessarily of a generalized character, which however appears to parallel the heuristic usage, in which T is involved only relative to an undetermined "vacuum-to-vacuum" amplitude, formally $\langle Tv, v\rangle$, which is generally ambiguous if not "infinite." For general $S \in Sp(\mathbf{H})$, the analog to the implementing unitary $\Gamma(S)$ that exists when S satisfies the Hilbert–Schmidt condition will map the domain $\mathbf{E}(H)$ of regular vectors outside of $\mathbf{K}$, into $^*\mathbf{E}(H)$, in which $\mathbf{K}$ is dense. Because of the difference in the domain and range of such an operator T, its defining equation is naturally taken in the form $TW(z) = W(Sz)T$, which is then interpreted as an equality of sesquilinear forms. We formalize the ambiguity regarding scalar factors that may be applied to such T by defining a given symplectic $S \in Sp(\mathbf{H})$ to be *projectively implementable* if there exists a continuous sesquilinear form T on $\mathbf{E}(H)$ such that

$$T(W(z)u, u') = T(u, W(Sz)^* u')$$

for all z in $E(B)$ and u and u' in $E(H)$, where in addition the non-triviality and normalization condition $T(v, v) = 1$ is imposed.

We first derive a kernel for the representation of $\Gamma(S)$ as an integral operator in the complex wave representation, in the case when $[S, i]$ is a Hilbert–Schmidt operator on $\mathbf{H}^{\#}$. Such a kernel has been given by Vergne [7], but it applies only to a subgroup, and the full treatment appears to be unpublished. Here a more general method is used, obtaining an explicit kernel for arbitrary symplectics satisfying the Hilbert–Schmidt restriction. On this subgroup, $\Gamma(S)$ is capable of normalization as a unitary operator by the constraint $\langle \Gamma(S)v, v\rangle > 0$, and with this constraint there is a corresponding cocycle, which is here given an explicit expression. The Hilbert–Schmidt restriction is then removed, obtaining projective rather than unitary implementability.

THEOREM 4. *All of $Sp(\mathbf{H})$ is projectively implementable. The form T implementing $S \in Sp(\mathbf{H})$ is projectively representable by a unique entire standard kernel $K_S(z, z')$ in the complex wave representation, having the form*

$$K_S(z, z') = \exp(\langle M(S)z, z\rangle + \langle z', M(S^{-1})z'\rangle + \langle z', N(S)z\rangle),$$

where

$$M(S) = \tfrac{1}{2}(SS^{\#} - I)(SS^{\#} + I)^{-1}; \qquad N(S) = (I - 2M(S^{-1}))\, S^{-1}.$$

LEMMA 4.1. *Let u be arbitrary in $E(B)$, and let V be arbitrary in $U(\mathbf{H})$. Let K be an arbitrary continuous sesquilinear form on $\mathbf{E}(H)$ of standard* *kernel $K(z, z')$. Then the standard kernels for the indicated forms have the following expressions*:

Form	Expression
$W(u)K$	$K(z-\sigma u, z')\exp(\sigma\langle u, z\rangle - \frac{1}{4}\langle u, u\rangle)$
$KW(u)$	$K(z, z'+\sigma u)\exp(-\sigma\langle z', u\rangle - \frac{1}{4}\langle u, u\rangle)$
$W(u)^* KW(u)$	$K(z+\sigma u, z'+\sigma u)\exp(-\sigma\langle u, z\rangle - \sigma\langle z', u\rangle - \frac{1}{2}\langle u, u\rangle)$
$\Gamma(V)K$	$K(V^*z, z')$
$K\Gamma(V)$	$K(z, Vz')$
$\Gamma(V)K\Gamma(V)^*$	$K(V^*z, V^*z')$

Proof. It will suffice to detail the derivation of the first expression, since the other expressions are similarly derivable, with appropriate treatment of the generalized integrals involved as indicated earlier. For arbitrary $f, g \in E(\mathbf{H})$,

$$
\begin{aligned}
(W(u)K)(f, g) &= K(f, W(-u)g) \\
&= \iint K(z, z')\, f(z')\, \overline{W(-u)\, g(z)}\, dv(z)\, dv(z') \\
&= \iint K(z, z')\, f(z')\, \bar{g}(z+\sigma u) \\
&\qquad \times \exp(-\sigma\langle z, u\rangle - \tfrac{1}{4}\langle u, u\rangle)\, dv(z)\, dv(z').
\end{aligned}
$$

With the variable of integration z changed to $y = z + \sigma u$ the integral becomes

$$
\begin{aligned}
&\iint K(y-\sigma u, z')\, f(z')\, \bar{g}(y)\exp(-\sigma\langle y-\sigma u, u\rangle - \tfrac{1}{4}\langle u, u\rangle) \\
&\qquad \times [dv(z)/dv(y)]\, dv(y)\, dv(z').
\end{aligned}
$$

Noting that $dv(z)/dv(y) = \exp(\sigma\langle y, u\rangle + \sigma\langle u, y\rangle - \frac{1}{2}|u|^2)$, the integral becomes

$$\iint K(y-\sigma u, z')\, f(z')\, \bar{g}(y)\exp(\sigma\langle u, y\rangle - \tfrac{1}{4}\langle u, u\rangle)\, dv(y)\, dv(z'),$$

from which the given expression follows. ∎

LEMMA 4.2. *Suppose that $\mathbf{H}$ is finite-dimensional, and that for some real part $\mathbf{H}'$ of $\mathbf{H}$, S takes the form $S(x+iy) = Rx + iR^{-1}y$ for $x, y \in \mathbf{H}'$, where R is a positive definite operator on $\mathbf{H}'$. Then S is unitarily implemented on*

K *by the product of the transformation having the kernel given by the theorem with the (unitarizing) factor,* $\sqrt{\det(2R(I+R^2)^{-1})}$.

Proof. By diagonalizing R, the proof can be reduced to the case when **H** is one-dimensional. Consider therefore the symplectic transformation $S: x+iy \to \lambda x + i\lambda^{-1}y$, $\lambda > 0$. It is readily checked that in the real wave representation (e.g., Segal [5]), in which **K** is represented as $L_2(\mathbb{R}, \nu_1)$, $\Gamma(S)$ is given as follows, in the normalization in which $\langle \Gamma(S)v, v\rangle > 0$:

$$f(x) \to \lambda^{-1/2} f(\lambda^{-1}u) \exp(-(\lambda^{-2}-1)\, u^2/4).$$

Using the intertwining relation between the real and complex wave representations [5], it follows that $\Gamma(S)$ takes the following form in the complex wave representation:

$$(\Gamma(S)F)(z) = \int_{\mathbb{C}} \sqrt{\frac{2\lambda}{1+\lambda^2}} \exp\left(\frac{\lambda^2-1}{2(\lambda^2+1)}\,\bar{z}^2 - \frac{\lambda^2-1}{2(\lambda^2+1)}\,w^2 + \frac{2\lambda}{(\lambda^2+1)}\,\bar{z}w\right) F(w)\, d\nu(w).$$

As a real matrix on $\mathbf{H}' \oplus \mathbf{H}'$, $S = \left(\begin{smallmatrix} R & 0 \\ 0 & R^{-1}\end{smallmatrix}\right)$, whence

$$M(S) = \begin{pmatrix} \frac{1}{2}(R^2-I)(R^2+I)^{-1} & 0 \\ 0 & -\frac{1}{2}(R^2-I)(R^2+I)^{-1} \end{pmatrix};$$

$$N(S) = \begin{pmatrix} 2R(I+R^2)^{-1} & 0 \\ 0 & 2R(I+R^2)^{-1} \end{pmatrix}.$$

In the case $\dim \mathbf{H} = 1$ with $R = \lambda$, this reproduces the indicated expression. ∎

LEMMA 4.3. *Suppose the symplectic S on* **H** *takes the form* $S(x+iy) = Rx + iR^{-1}y$ $(x, y \in \mathbf{H}')$ *for some real part* $\mathbf{H}'$ *of* **H**, *where R is positive definite on* $\mathbf{H}'$ *and $R-I$ is Hilbert–Schmidt. Then S is unitarily implemented on* **K** *by the operator $\Gamma(S)$, whose kernel in the complex wave representation is*

$$K_S(z, z') = (\det(2(I+SS^{\#})^{-1}))^{1/4} \times \exp(\langle M(S)z, z\rangle + \langle z', M(S^{-1})\, z'\rangle + \langle z', N(S)z\rangle).$$

Proof. Let $P_1, P_2, \ldots$ be a sequence of projections of finite rank on $\mathbf{H}'$ that commute with R and are such that $(R-I)\,P_j$ converges to $R-I$ in the Hilbert–Schmidt norm as $j \to \infty$. Set $R_j = I + (R-I)\,P_j$. Then by

Lemma 4.2, S_j, where $S_j z = R_j x + iR_j^{-1} y$ for $x, y \in \mathbf{H}'$, is unitarily implementable on $\mathbf{K}$ with kernel

$$[\det(2(I + S_j S_j^{\#})^{-1})]^{1/4}$$
$$\times \exp(\langle M(S_j) z, z \rangle + \langle z', M(S_j^{-1}) z' \rangle + \langle z', N(S_j)) z \rangle).$$

It is not difficult to verify that the map $R' \to \det[2R'(I + R'^2)^{-1}]$ from the positive definite operators R' on $\mathbf{H}'$ of the form $R' = I + X$, where X is Hilbert–Schmidt, to the reals, is well defined and continuous in the Hilbert–Schmidt norm on X. It follows that $\det[2(I + S_j S_j^{\#})^{-1}] \to \det[2(I + SS^{\#})^{-1}]$ as $j \to \infty$. It is also readily verified that $M(S_j) \to M(S)$ and $N(S_j) \to N(S)$ in the strong operator topology on $\mathbf{H}^{\#}$. It follows that $K_S(z, z') = \lim_j K_{S_j}(z, z')$, which has the form claimed in Lemma 4.3. ∎

Proof of Theorem. By the polar decomposition of symplectics (e.g., Shale [6]), the given S has the form UG, where U is unitary on $\mathbf{H}$ and $G = \sqrt{SS^{\#}}$. For a suitable real part $\mathbf{H}'$ of $\mathbf{H}$, G has the form $x + iy \to Rx + iR^{-1}y$, where R is an invertible positive self-adjoint operator on $\mathbf{H}'$. For any such operator R, there exists a sequence R_n of similar operators that is convergent to R in the strong operator topology, and such that $R_n - I$ has finite-dimensional range. By the lemmas, the conclusion of Theorem 4 is applicable to the corresponding operators G_n: $x + iy \to R_n x + iR_n^{-1} y$. As $n \to \infty$, $K_{G_n}(z, z') \to K_G(z, z')$, and the estimates obtained in Section 3 show that $T_n(u, u') \to T(u, u')$, where T_n and T are the forms corresponding to K_{G_n} and to K_G, for arbitrary $u, u' \in E(H)$. Thus T is representable by a standard kernel, as claimed in the theorem.

By Lemma 4.1, $\Gamma(U) K_G$ has the kernel $K_G(U^* z, z')$. On the other hand, evidently $\Gamma(U) K_G$ projectively implements UG, if K_G implements G. It follows that kernel $K_G(U^* z, z')$ represents the form projectively implementing S. The proof is concluded by observing that this kernel coincides with that given by the theorem. ∎

The unitary implementability of symplectics in the subgroup $Sp_2(\mathbf{H})$ of $Sp(\mathbf{H})$ of symplectics that satisfy the Hilbert–Schmidt constraint defines a cocycle, which derives from the uniqueness of the implementing unitary T within a phase factor. We define the cocycle $c(S', S)$ by the equation $\Gamma(S') \Gamma(S) = c(S', S) \Gamma(S'S)$, where $\Gamma(S)$ is normalized by the condition that $\langle \Gamma(S) v, v \rangle > 0$, which fixes the otherwise ambiguous phase. This agrees with the usual normalization on the subgroup $U(\mathbf{H})$, on which c is identically 1.

In order to avoid confusion between the complex number i and the complex structure in $\mathbf{H}$, we denote the latter as J. Thus, relative to any real part $\mathbf{H}'$ of $\mathbf{H}$ and the corresponding representation $\mathbf{H} = \mathbf{H}' \oplus \mathbf{H}'$ via the identification of the vector $x+iy$ ($x, y \in \mathbf{H}'$) with $x \oplus y$, J has the matrix $\begin{pmatrix} 0 & -I' \\ I' & 0 \end{pmatrix}$, where I' is the identity operator on $\mathbf{H}'$. In these terms we may state

COROLLARY 4.1. *For arbitrary S' and S in $Sp_2(\mathbf{H})$,*

$$c(S', S) = [\det(\tfrac{1}{2}(I+S'S'^{\#})(I+SS^{\#})(I+(S'S)(S'S)^{\#})^{-1})]^{-1/4}$$
$$\times [\det(I - M(S) - M(S'^{-1}) + iJ(M(S) - M(S'^{-1})))]^{-1/2}.$$

Proof. By definition, $c(S', S) = \langle \Gamma(S')\Gamma(S)v, v\rangle / \langle \Gamma(S'S)v, v\rangle$. Using Lemma 4.3,

$$c(S', S) = \left[\det\left(\frac{1}{2}(I+S'S'^{\#})(I+SS^{\#})(I+(S'S)(S'S)^{\#})^{-1}\right)\right]^{-1/4}$$
$$\times \int \exp(\langle M(S)w, w\rangle + \langle w, M(S'^{-1})w\rangle)\, dv(w). \tag{5}$$

Since $\langle \Gamma(R)v, v\rangle > 0$ for all $R \in Sp_2(\mathbf{H})$, $c(S', S)$ differs from $\langle \Gamma(S'S)v, v\rangle$ only by the first three factors, each of the form $\det(\frac{1}{2}(I+RR^{\#}))$ for suitable $R \in Sp_2(\mathbf{H})$, in the expression given in the theorem. This factor $f(S', S)$ is given in terms of the kernels for S and S' as

$$f(S', S) = \iiint K_{S'}(z, w)\, K_S(w, z')\, dv(z)\, dv(z')\, dv(w)$$
$$= \int \exp(\langle M(S')w, w\rangle + \langle w, M(S^{-1})w\rangle)\, dv(w).$$

To complete the proof, it suffices, using approximation by finite-dimensional spaces, to establish

LEMMA 4.4. *Let A and B be bounded antilinear operators on $\mathbf{H}$ that are self-adjoint on $\mathbf{H}^{\#}$. Suppose $\|A\| < 1$ and $\|B\| < 1$. Then*

$$\int \exp(\langle Aw, w\rangle + \langle w, Bw\rangle)\, dv(w) = [\det(I - A - B + iJ(A-B))]^{-1/2}.$$

Proof. Since $A = A^{\#}$ and A is antilinear, there exists a real part $\mathbf{H}'$ of $\mathbf{H}$ such that $A(x+iy) = A_1 x - iA_1 y$ for $x, y \in \mathbf{H}'$, where A_1 is bounded self-adjoint on $\mathbf{H}'$. The antilinearity of B implies that it has the matrix $\begin{pmatrix} B_1 & B_2 \\ B_2 & -B_1 \end{pmatrix}$ relative to the representation of $\mathbf{H}$ as $\mathbf{H}' \oplus \mathbf{H}'$. In terms of this representation, it follows that

$$\int \exp(\langle Aw, w\rangle + \langle w, Bw\rangle)\, d\nu(w)$$

$$= \iint \exp(\langle D(x \oplus y), (x \oplus y)\rangle_{\mathbf{H}' \oplus \mathbf{H}'})\, dx\, dy$$

$$= [\det(D)]^{-1/2}$$

$$= [\det(I - A - B + iJ(A - B))]^{-1/2},$$

where

$$D = \begin{pmatrix} I - A_1 - B_1 + iB_2 & i(A_1 - B_1) - B_2 \\ i(A_1 - B_1) - B_2 & I + A_1 + B_1 - iB_2 \end{pmatrix}. \quad \blacksquare$$

The space $E(H)$ is of almost maximal regularity from the standpoint of applications. Depending on the application, spaces of almost minimal regularity may on occasion be more appropriate. In particular, the use of entire vectors in the present connection can be replaced by the use of analytic vectors.

COROLLARY 4.2. *The conclusion of Theorem* 4 *remains valid if the projective implementability is redefined to mean the existence of a continuous form T on $\mathbf{A}(H)$ such that $T(W(z)u, u') = T(u, W(z)^* u')$ and $T(v, v) = 1$ for all $u, u' \in A(H)$ and $z \in A(B)$.*

Proof. The essential observation on which a straightforward adaptation of the proof of Theorem 4 is based is that the conclusion that $K_{G_n}(z, z') \to K_G(z, z')$ and appropriate estimates from Section 3 remain valid when the regularity obtained by replacing z and z' by $\exp(-tB)z$ and $\exp(-tB)z'$ for arbitrary large t, which is available in connection with the space $\mathbf{E}(H)$, remains applicable also if t is sufficiently small, in the context of $\mathbf{A}(H)$. The proof is otherwise the same and further details are omitted. $\blacksquare$

Remark 1. An expression for the second factor $\tilde{c}(S', S)$ in Eq. (5) that is manifestly (not only de facto) independent of the choice of $\mathbf{H}'$ can be derived from an observation due to M. Vergne. Denoting the similarity of matrices as $\sim$,

$$D \sim \begin{pmatrix} I - 2A_1 & i(I + 2A_1) \\ i(A_1 - B_1) - B_2 & I + A_1 + B_1 - iB_2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 2I & i(I + 2A_1) \\ -i(I - 2iB_1) - 2B_2 & I + A_1 + B_1 - iB_2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 2I & 0 \\ -i(I - 2iB_1) - 2B_2 & \frac{1}{2}(I - 4A_1(B_1 - iB_2)) \end{pmatrix}.$$

The determinant of the latter matrix is just the determinant of $I-4AB$ as an operator on the complex Hilbert space **H**. Accordingly

$$\tilde{c}(S', S) = \det(I - 4M(S)\, M(S'^{-1}))$$

(note that $4M(S)\, M(S'^{-1})$ is a linear operator on **H**, as the product of two antilinear operators).

Remark 2. The question of essentiality of the cocycle and the ambiguity in its sign is an abstract form of the explication of the so-called Schwinger term in physical quantum field theory. We plan to treat this matter elsewhere.

Remark 3. In some applications, the symplectic operator S is not even bounded (e.g., von Neumann [8, under Light-Theory]). There is no essential difficulty in adapting the present method to the development of projective implementability for such transformations, in appropriate ad hoc spaces of regularity, albeit at the cost of intensive specialization and loss of covariance.

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